

Vector fields and foliations associated to groups of projective automorphisms

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Abstract

We introduce and give normal forms for (one-dimensional) Riccati foliations (vector fields) on $\overline{\mathbb{C}} \times \mathbb{C}P(2)$ and $\overline{\mathbb{C}} \times \overline{\mathbb{C}}^n$. These foliations are characterized by transversality with the generic fiber of the first projection and we prove they are conjugate *in some invariant Zariski open subset* to the suspension of a group of automorphisms of the fiber, $\mathbb{C}P(2)$ or $\overline{\mathbb{C}}^n$, this group called *global holonomy*. Our main result states that given a finitely generated subgroup G of $\text{Aut}(\mathbb{C}P(2))$, there is a Riccati foliation on $\overline{\mathbb{C}} \times \mathbb{C}P(2)$ for which the global holonomy is conjugate to G .

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1 Introduction

Foliations transverse to fibrations are among the simplest (constructive) examples of foliated manifolds, once regarded as suspensions of group of diffeomorphisms ([2], [4]). Thus one expects to perform a nice study of them in the global theoretic aspect. In the complex algebraic setting, foliations usually exhibit singularities so this possibility cannot be excluded. Very representative examples of either situations are given by the class of *Riccati foliations* ([7]) in dimension two. A very interesting study is performed in [11] and a complete reference on the two dimensional case is [10]. In this paper we study one-dimensional holomorphic foliations with singularities which are transverse to a given holomorphic fibration off some exceptional set in a sense that we shall make precise below. Let us first recall the classical notion. Let $\eta = (E, \pi, B, F)$ be a fibre bundle with total space E , fiber F , basis B and projection $\pi: E \rightarrow B$. A foliation \mathcal{F} on E is *transverse to η* if: (1) for each $p \in E$, the leaf L_p of \mathcal{F} with $p \in L_p$ is transverse to the fiber $\pi^{-1}(q)$, $q = \pi(p)$; (2) $\dim(\mathcal{F}) + \dim(F) = \dim(E)$; and (3) for each leaf L of \mathcal{F} , the restriction $\pi|_L: L \rightarrow B$ is a covering map. According to Ehresman ([4]) if the fiber F is compact, then the conditions (1) and (2) already

imply (3). In the complex setting all the objects above are holomorphic by hypothesis and several are the interesting frameworks (see [14] for the complex hyperbolic case). Under the presence of singularities, a weaker notion must be introduced. We shall say that \mathcal{F} is *transverse to almost every fiber* of the fibre bundle η if there is an analytic subset $\Lambda(\mathcal{F}) \subset E$ which is union of fibers of η , such that the restriction \mathcal{F}_0 of \mathcal{F} to $E_0 = E \setminus \Lambda$ is transverse to the natural subbundle η_0 of η having E_0 as total space. If $\Lambda(\mathcal{F})$ is minimal with this property then $\Lambda(\mathcal{F})$ is called the *exceptional set* of \mathcal{F} . By a *Riccati foliation* we mean a foliation \mathcal{F} as above, for which the exceptional set $\Lambda(\mathcal{F})$ is \mathcal{F} -invariant. In particular we shall consider the *global holonomy* of \mathcal{F} as the global holonomy of the restriction \mathcal{F}_0 on $E_0 = E \setminus \Lambda(\mathcal{F})$. In the classical situation of Riccati foliations in $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$, the global holonomy is a finitely generated group of Möbius transformations, *i.e.*, a finitely generated subgroup of $PSL(2, \mathbb{C})$. Using the well-known classification of Möbius maps by the set of fixed points (see Beardon [1]), Lins Neto is able to prove ([7]) that given a finitely generated subgroup $G < PSL(2, \mathbb{C})$ there is a Riccati foliation in $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$ for which the global holonomy is conjugated to G . Similarly, in this work for the case of Riccati foliations on $\overline{\mathbb{C}} \times \mathbb{C}P(2)$, the study of the global holonomy relies on the classification of holomorphic diffeomorphisms of $\mathbb{C}P(2)$ by the set of fixed points. This is the content of Theorem 4 which applies to the problem of construction of foliations on $\overline{\mathbb{C}} \times \mathbb{C}P(2)$ with given global holonomy group:

Theorem 1 (Synthesis theorem). *Let $x_0, x_1, \dots, x_k \in \overline{\mathbb{C}}$ be points. Let $f_1, \dots, f_k \in \text{Aut}(\mathbb{C}P(2))$ be biholomorphisms. Then there is a Riccati foliation \mathcal{F} on $\overline{\mathbb{C}} \times \mathbb{C}P(2)$ such that the invariant fibers of \mathcal{F} are $\{x_0\} \times \mathbb{C}P(2), \dots, \{x_k\} \times \mathbb{C}P(2)$ and the global holonomy of \mathcal{F} is conjugate to the subgroup of $\text{Aut}(\mathbb{C}P(2))$ generated by f_1, \dots, f_k .*

As mentioned above, our basic motivation comes from the classical Riccati foliations in dimension two, *i.e.*, Riccati foliations on $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$, such foliations being given in affine coordinates by polynomial vector fields of the form $X(x, y) = p(x) \frac{\partial}{\partial x} + \left(a(x)y^2 + b(x)y + c(x) \right) \frac{\partial}{\partial y}$. In this Riccati case the fiber is a (compact) rational Riemann sphere and the exceptional set $\Lambda \subset \overline{\mathbb{C}} \times \overline{\mathbb{C}}$ is a finite union of vertical projective lines $\{x\} \times \overline{\mathbb{C}}$ and is invariant by the foliation. In this paper we shall mainly work with singular holomorphic foliations \mathcal{F} on $\overline{\mathbb{C}} \times M$ where $M = \overline{\mathbb{C}}^n$ or $M = \mathbb{C}P(2)$, which are transverse to almost every fiber of η , with projection $\pi : \overline{\mathbb{C}} \times M \rightarrow \overline{\mathbb{C}}$, $(x, y) \mapsto x$. For the case $M = \overline{\mathbb{C}}^n$ these foliations have a natural normal form like in the Riccati case as follows:

Theorem 2. *Let \mathcal{F} be a singular holomorphic foliation on $\overline{\mathbb{C}} \times \overline{\mathbb{C}}^n$ given by a polynomial vector field X in affine coordinates $(x, y) \in \mathbb{C} \times \mathbb{C}^n$. Suppose \mathcal{F} is transverse to almost every fiber of the bundle η where $\pi : \overline{\mathbb{C}} \times \overline{\mathbb{C}}^n \rightarrow \overline{\mathbb{C}}$, $\pi(z_1, z_2) = z_1$ is the projection of η . Then \mathcal{F} is a Riccati foliation and $X(x, y) = p(x) \frac{\partial}{\partial x} + (q_{1,2}(x)y_1^2 + q_{1,1}(x)y_1 + q_{1,0}(x)) \frac{\partial}{\partial y_1} + \dots + (q_{n,2}(x)y_n^2 + q_{n,1}(x)y_n + q_{n,0}(x)) \frac{\partial}{\partial y_n}$.*

For the case $M = \mathbb{C}P(2)$ the classification does not follow an already established model. Indeed, owing to Okamoto [9], given $a, b \in \mathbb{C}$ the vector field $X(x, y, z) = \frac{\partial}{\partial x} + (z - y^2) \frac{\partial}{\partial y} - (a + by + yz) \frac{\partial}{\partial z}$ induces a foliation in $\overline{\mathbb{C}} \times \mathbb{C}P(2)$ which is transverse to every fiber $\{x\} \times \mathbb{C}P(2)$ except for $x = \infty$. Our normal form, englobing this class of examples, is as follows:

Theorem 3. *Let \mathcal{F} be a singular holomorphic foliation on $\overline{\mathbb{C}} \times \mathbb{C}P(2)$ given by a polynomial vector field X in affine coordinates (x, y, z) on $\mathbb{C} \times \mathbb{C}^2$. If \mathcal{F} is transverse to almost every fiber of the fibre bundle η where $\pi : \overline{\mathbb{C}} \times \mathbb{C}P(2) \rightarrow \overline{\mathbb{C}}$, $\pi(z_1, z_2) = z_1$ is the projection of η , then \mathcal{F} is a Riccati*

foliation and $X(x, y, z) = p(x)\frac{\partial}{\partial x} + Q(x, y, z)\frac{\partial}{\partial y} + R(x, y, z)\frac{\partial}{\partial z}$; in affine coordinates $(x, y, z) \in \mathbb{C} \times \mathbb{C}^2 \hookrightarrow \overline{\mathbb{C}} \times \mathbb{C}P(2)$ where

$$Q(x, y, z) = A(x) + B(x)y + C(x)z + D(x)yz + E(x)y^2,$$

$$R(x, y, z) = a(x) + b(x)y + c(x)z + E(x)yz + D(x)y^2,$$

and $p, a, b, c, A, B, C, D, E \in \mathbb{C}[x]$.

Acknowledgement: We are grateful to R. S. Mol, Julio Canille and L. G. Mendes for reading the original manuscript and various valuable suggestions.

2 Classification of automorphisms of $\mathbb{C}P(2)$

The group of automorphisms of $\mathbb{C}P(n)$ is induced by the general linear group, that is, $\text{Aut}(\mathbb{C}P(n)) \cong \text{PGL}(n+1, \mathbb{C})$ ([5]), it identifies an isomorphism $T: \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{n+1}$ with the biholomorphism of the complex projective space $[T]$ defined by: if $r \subset \mathbb{C}^{n+1}$ is a complex line contains $0 \in \mathbb{C}^{n+1}$, then $s = T(r)$ is a complex line contains $0 \in \mathbb{C}^{n+1}$ and we consider $[T]: \mathbb{C}P(n) \rightarrow \mathbb{C}P(n)$ given by $[T](r \setminus \{0\}) = s \setminus \{0\}$.

Aiming the study of Riccati foliations on $\overline{\mathbb{C}} \times \mathbb{C}P(2)$ through the global holonomy we perform the classification of holomorphic diffeomorphisms of $\mathbb{C}P(2)$ by the set of fixed points. This the content of the following result:

Theorem 4. *If $f: \mathbb{C}P(2) \rightarrow \mathbb{C}P(2)$ is a biholomorphism and $\Sigma(f)$ denotes its set of fixed points then we have the following six possibilities:*

1. $\Sigma(f)$ has pure dimension zero and is a set of one, two or three points.
2. $\Sigma(f)$ has pure dimension one and consists of two projective lines.
3. $\Sigma(f)$ consists of one point and two projective lines.
4. $\Sigma(f)$ has dimension two and $\Sigma(f) = \mathbb{C}P(2)$.

In particular, f is conjugate in $\text{Aut}(\mathbb{C}P(2))$ to a map $g \in \text{Aut}(\mathbb{C}P(2))$ of the form $g(x : y : z) = (\lambda_0 x + y : \lambda_0 y + z : \lambda_0 z)$, $g(x : y : z) = (\lambda_0 x + y : \lambda_0 y : \lambda_1 z)$, $g(x : y : z) = (\lambda_0 x : \lambda_1 y : \lambda_2 z)$, $g(x : y : z) = (\lambda_0 x : \lambda_0 y : \lambda_1 z)$, $g(x : y : z) = (\lambda_0 x + y : \lambda_0 y : \lambda_0 z)$, $g(x : y : z) = (x : y : z)$, where $\lambda_0, \lambda_1, \lambda_2 \in \mathbb{C} \setminus \{0\}$, respectively.

Proof of Theorem 4. If $f \in \text{Aut}(\mathbb{C}P(2))$, then there is $A = (a_{ij})_{3 \times 3} \in \text{GL}(3, \mathbb{C})$ such that $f = [A]$, that is, $f(x : y : z) = (a_{11}x + a_{12}y + a_{13}z : a_{21}x + a_{22}y + a_{23}z : a_{31}x + a_{32}y + a_{33}z)$. We use the Jordan canonical forms and obtain the classification of automorphisms of $\mathbb{C}P(2)$ by fixed points. In fact, there are three possibilities for the characteristic polynomial of A , $p_A(t)$, in $\mathbb{C}[t]$:

- (i) $p_A(t) = (t - \lambda_0)(t - \lambda_1)(t - \lambda_2)$;
- (ii) $p_A(t) = (t - \lambda_0)^2(t - \lambda_1)$;

$$(iii) \ p_A(t) = (t - \lambda_0)^3,$$

where $\lambda_0, \lambda_1, \lambda_2 \in \mathbb{C} \setminus \{0\}$ are different.

Case (i). The minimal polynomial of A , $m_A(t)$, in $\mathbb{C}[t]$ is $m_A(t) = p_A(t)$. Then there is $P \in \text{GL}(3, \mathbb{C})$ such that $A = P^{-1}JP$ where

$$J = \begin{bmatrix} \lambda_0 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

is the Jordan canonical form of A . Therefore f is conjugate to $[J]$ because $f = [A] = [P^{-1}][J][P]$. We consider $g \equiv [J]$, that is, $g : \mathbb{C}P(2) \rightarrow \mathbb{C}P(2)$ defined by $g(x : y : z) = (\lambda_0 x : \lambda_1 y : \lambda_2 z)$ with $\lambda_0, \lambda_1, \lambda_2 \in \mathbb{C} \setminus \{0\}$. We shall determinate the fixed points of g . First, we recall that $\mathbb{C}P(2)$ is a complex manifold defined by the atlas $\{(E_j, \varphi_j)\}_{j \in \{0,1,2\}}$ where

$$E_j = \{(z_0 : z_1 : z_2) \in \mathbb{C}P(2); z_j \neq 0\},$$

and $\varphi_j : E_j \rightarrow \mathbb{C}^2$ is defined by $\varphi_0(z_0 : z_1 : z_2) = \left(\frac{z_1}{z_0}, \frac{z_2}{z_0}\right)$, $\varphi_1 : E_1 \rightarrow \mathbb{C}^2$, $\varphi_1(z_0 : z_1 : z_2) = \left(\frac{z_0}{z_1}, \frac{z_2}{z_1}\right)$, $\varphi_2 : E_2 \rightarrow \mathbb{C}^2$, $\varphi_2(z_0 : z_1 : z_2) = \left(\frac{z_0}{z_2}, \frac{z_1}{z_2}\right)$. Observe that f is conjugate to g so f has the same numbers of fixed points that g . Now we obtain the points fixed by g . First, we consider the points $(x : y : 1) \in \mathbb{C}P(2)$. In this case we have $g(x : y : 1) = (\lambda_0 x : \lambda_1 y : \lambda_2)$ and the application $G : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ defined by $G(x, y) = \left(\frac{\lambda_0}{\lambda_2}x, \frac{\lambda_1}{\lambda_2}y\right)$. We obtain the following commutative diagram:

$$\begin{array}{ccc} E_2 & \xrightarrow{f} & E_2 \\ \varphi_2 \downarrow & & \downarrow \varphi_2 \\ \mathbb{C}^2 & \xrightarrow{G} & \mathbb{C}^2 \end{array}$$

Therefore the fixed points of $g \in \text{Aut}(\mathbb{C}P(2))$ of the form $(x : y : 1)$ are given by the solutions of the following system

$$\begin{cases} \frac{\lambda_0}{\lambda_2}x &= x \\ \frac{\lambda_1}{\lambda_2}y &= y \end{cases}$$

and the point $(0, 0)$ is this solution so $(0 : 0 : 1)$ is a fixed point by g . By analogy with it we consider the points of the following form $(x : 1 : z) \in \mathbb{C}P(2)$. Now we have $g(x : 1 : z) = (\lambda_0 x : \lambda_1 : \lambda_2 z)$ and $G : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ defined by $G(x, z) = \left(\frac{\lambda_0}{\lambda_1}x, \frac{\lambda_2}{\lambda_1}z\right)$ such that the following diagram is commutative:

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_1 \\ \varphi_1 \downarrow & & \downarrow \varphi_1 \\ \mathbb{C}^2 & \xrightarrow{G} & \mathbb{C}^2 \end{array}$$

Notice that the fixed points of G are given by the solutions of the system

$$\begin{cases} x &= \frac{\lambda_0}{\lambda_1}x \\ z &= \frac{\lambda_2}{\lambda_1}z \end{cases}$$

On the other hand this system have the solution $(0,0)$ only. Therefore $(0 : 1 : 0)$ is another fixed point by g . And we consider the points of the following form $(1 : y : z) \in \mathbb{C}P(2)$, too. We obtain $g(1 : y : z) = (\lambda_0 : \lambda_1 y : \lambda_2 z)$ and $G : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ defined by $G(y, z) = \left(\frac{\lambda_1}{\lambda_0} y, \frac{\lambda_2}{\lambda_0} z \right)$ such that the following diagram is commutative:

$$\begin{array}{ccc} E_0 & \xrightarrow{f} & E_0 \\ \varphi_0 \downarrow & & \downarrow \varphi_0 \\ \mathbb{C}^2 & \xrightarrow{F} & \mathbb{C}^2 \end{array}$$

And the fixed points of G are given by the solutions of the following system

$$\begin{cases} y &= \frac{\lambda_1}{\lambda_0} y \\ z &= \frac{\lambda_2}{\lambda_0} z \end{cases}$$

Now we have the point $(0,0)$ the only solution. Then $(1 : 0 : 0)$ is a fixed point by g . Therefore the fixed points of g are

$$\text{Fix}(g) = \{(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)\}.$$

By analogy with these ideas in the other cases we obtain:

Case (ii). There are two possibilities for the minimal polynomial of A , $m_A(t)$, in $\mathbb{C}[t]$:

$$(ii.1) \quad m_A(t) = (t - \lambda_0)(t - \lambda_1);$$

$$(ii.2) \quad m_A(t) = (t - \lambda_0)^2(t - \lambda_1) = p_A(t),$$

where $\lambda_0, \lambda_1 \in \mathbb{C} \setminus \{0\}$, $\lambda_0 \neq \lambda_1$. In both of them there is $P \in \text{GL}(3, \mathbb{C})$ such that $A = P^{-1}JP$ where J is the Jordan canonical form of A . Then $f = [A] = [P^{-1}][J][P]$, that is, f is conjugate to $[J]$. Therefore

Case (ii.1). In this case we have

$$J = \begin{bmatrix} \lambda_0 & 0 & 0 \\ 0 & \lambda_0 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix}.$$

and then f is conjugate to $g \equiv [J]$, that is, $g(x : y : z) = (\lambda_0 x : \lambda_0 y : \lambda_1 z)$. Let us study the fixed points of g . We consider first the points the following form: $(x : y : 1) \in \mathbb{C}P(2)$. We obtain $g(x : y : 1) = (\lambda_0 x : \lambda_0 y : \lambda_1)$ and $G : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ defined by $G(x, y) = \left(\frac{\lambda_0}{\lambda_1} x, \frac{\lambda_0}{\lambda_1} y \right)$ such that the following diagram is commutative:

$$\begin{array}{ccc} E_2 & \xrightarrow{g} & E_2 \\ \varphi_2 \downarrow & & \downarrow \varphi_2 \\ \mathbb{C}^2 & \xrightarrow{G} & \mathbb{C}^2 \end{array}$$

Then the fixed points of G are the solutions the following system

$$\begin{cases} \frac{\lambda_0}{\lambda_1} x &= x \\ \frac{\lambda_0}{\lambda_1} y &= y \end{cases}$$

and note that $(0, 0)$ is this solution. Therefore $(0 : 0 : 1)$ is fixed point by $g \in \text{Aut}(\mathbb{C}P(2))$. Now we consider the points the following form: $(x : 1 : z) \in \mathbb{C}P(2)$. We obtain $g(x : 1 : z) = (\lambda_0 x : \lambda_0 : \lambda_1 z)$ and $G : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ defined by $G(x, z) = \left(x, \frac{\lambda_1}{\lambda_0} z\right)$ such that the following diagram is commutative:

$$\begin{array}{ccc} E_1 & \xrightarrow{g} & E_1 \\ \varphi_1 \downarrow & & \downarrow \varphi_1 \\ \mathbb{C}^2 & \xrightarrow{G} & \mathbb{C}^2 \end{array}$$

The fixed points of G are given by the solutions the following system

$$\begin{cases} x &= x \\ z &= \frac{\lambda_1}{\lambda_0} z \end{cases}$$

and we have the following solutions: $\{(x, 0) \in \mathbb{C}^2; x \in \mathbb{C}\}$. Therefore $\text{Fix}_2(g) = \{(x : 1 : 0) \in \mathbb{C}P(2); x \in \mathbb{C}\}$ are fixed points of g . At the end we consider the points of the form $(1 : y : z) \in \mathbb{C}P(2)$. Then we have $g(1 : y : z) = (\lambda_0 : \lambda_0 y : \lambda_1 z)$ and $G : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ defined by $G(y, z) = \left(y, \frac{\lambda_1}{\lambda_0} z\right)$ such that

$$\begin{array}{ccc} E_0 & \xrightarrow{g} & E_0 \\ \varphi_0 \downarrow & & \downarrow \varphi_0 \\ \mathbb{C}^2 & \xrightarrow{G} & \mathbb{C}^2 \end{array}$$

commute. The fixed points of G are the solutions of the following system

$$\begin{cases} y &= y \\ z &= \frac{\lambda_1}{\lambda_0} z \end{cases} ,$$

that is, the points $\{(y, 0) \in \mathbb{C}^2; y \in \mathbb{C}\}$. Therefore the points $\text{Fix}_3(g) = \{(1 : y : 0) \in \mathbb{C}P(2); y \in \mathbb{C}\}$ are fixed by g , too. Then in this case the fixed points of g are two projective lines $\text{Fix}_2(g)$ and $\text{Fix}_3(g)$ and one point $(0 : 0 : 1) \in \mathbb{C}P(2)$.

Case (ii.2). In this case we obtain

$$J = \begin{bmatrix} \lambda_0 & 1 & 0 \\ 0 & \lambda_0 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix}$$

and f is conjugate by $g = [J]$, that is, $g(x : y : z) = (\lambda_0 x + y : \lambda_0 y : \lambda_1 z)$. Let us study of the fixed points of g . First, we consider the points of the form $(x : y : 1) \in \mathbb{C}P(2)$. Then we have $g(x : y : 1) = (\lambda_0 x + y : \lambda_0 y : \lambda_1)$ and $G : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ defined by $G(x, y) = \left(\frac{\lambda_0}{\lambda_1} x + \frac{1}{\lambda_1} y, \frac{\lambda_0}{\lambda_1} y\right)$ such that the following diagram is commutative

$$\begin{array}{ccc} E_2 & \xrightarrow{g} & E_2 \\ \varphi_2 \downarrow & & \downarrow \varphi_2 \\ \mathbb{C}^2 & \xrightarrow{G} & \mathbb{C}^2 \end{array}$$

The fixed points of G are the solutions of the following system

$$\begin{cases} \frac{\lambda_0}{\lambda_1}x + \frac{1}{\lambda_1}y = x \\ \frac{\lambda_0}{\lambda_1}y = y \end{cases}$$

and this is the point $(0,0) \in \mathbb{C}^2$ only. Then $(0 : 0 : 1) \in \mathbb{C}P(2)$ is a fixed point by $g \in \text{Aut}(\mathbb{C}P(2))$. On the other hand we consider the points of the following form $(x : 1 : z) \in \mathbb{C}P(2)$. We obtain $g(x : 1 : z) = (\lambda_0 x + 1 : \lambda_0 : \lambda_1 z)$ and $G : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ defined by $G(x, z) = \left(x + \frac{1}{\lambda_0}, \frac{\lambda_1}{\lambda_0}z\right)$ such that

$$\begin{array}{ccc} E_1 & \xrightarrow{g} & E_1 \\ \varphi_1 \downarrow & & \downarrow \varphi_1 \\ \mathbb{C}^2 & \xrightarrow{G} & \mathbb{C}^2 \end{array}$$

commute. Notice that the fixed points of G are the solutions of the following system

$$\begin{cases} x = x + \frac{1}{\lambda_0} \\ z = \frac{\lambda_1}{\lambda_0}z \end{cases}$$

Observe that there are not solutions of this system, then there are not fixed points of $g \in \text{Aut}(\mathbb{C}P(2))$ of the form $(x : 1 : z) \in \mathbb{C}P(2)$. Now we consider points of the form $(1 : y : z) \in \mathbb{C}P(2)$. We obtain $g(1 : y : z) = (\lambda_0 + y : \lambda_0 y : \lambda_1 z)$ and if $y \neq -\lambda_0$, then we have $G : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ defined by $G(y, z) = \left(\frac{\lambda_0 y}{\lambda_0 + y}, \frac{\lambda_1 z}{\lambda_0 + y}\right)$ such that the following diagram is commutative

$$\begin{array}{ccc} E_0 & \xrightarrow{g} & E_0 \\ \varphi_0 \downarrow & & \downarrow \varphi_0 \\ \mathbb{C}^2 & \xrightarrow{G} & \mathbb{C}^2 \end{array}$$

Now note that if $y = -\lambda_0$ then there are not fixed points of g . The fixed points of G are given by the solutions of the system

$$\begin{cases} y = \frac{\lambda_0 y}{\lambda_0 + y} \\ z = \frac{\lambda_1 z}{\lambda_0 + y} \end{cases}$$

that is, $(0,0)$ is the fixed point by G . Therefore $(1 : 0 : 0)$ is a fixed points of g are $(1 : 0 : 0)$ e $(0 : 0 : 1)$.

Case (iii). There are three possibilities for the minimal polynomial of A in $\mathbb{C}[t]$:

(iii.1) $m_A(t) = t - \lambda_0$;

(iii.2) $m_A(t) = (t - \lambda_0)^2$;

(iii.3) $m_A(t) = (t - \lambda_0)^3 = p_A(t)$,

where $\lambda_0 \in \mathbb{C} \setminus \{0\}$. In all possibilities there is $P \in \text{GL}(3, \mathbb{C})$ such that $A = P^{-1}JP$ where J is the Jordan canonical form of A . Then $f = [J] = [P^{-1}][J][P]$ is conjugate to $[J]$.

Case (iii.1). In this case we obtain

$$J = \begin{bmatrix} \lambda_0 & 0 & 0 \\ 0 & \lambda_0 & 0 \\ 0 & 0 & \lambda_0 \end{bmatrix}$$

and then $g = [J]$, that is, $g(x : y : z) = (\lambda_0 x : \lambda_0 y : \lambda_0 z)$. Therefore g is the identity application and all $\mathbb{C}P(2)$ are fixed by g .

Case (iii.2) In this case we have

$$J = \begin{bmatrix} \lambda_0 & 1 & 0 \\ 0 & \lambda_0 & 0 \\ 0 & 0 & \lambda_0 \end{bmatrix}$$

and then f is conjugate to $g(x : y : z) = (\lambda_0 x + y : \lambda_0 y : \lambda_0 z)$. Let us study the fixed points of g . We consider first the points the following form: $(x : y : 1) \in \mathbb{C}P(2)$. We obtain $g(x : y : 1) = (\lambda_0 x + y : \lambda_0 y : \lambda_0)$ and $G : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ defined by $G(x, y) = \left(x + \frac{1}{\lambda_0}y, y\right)$ such that

$$\begin{array}{ccc} E_2 & \xrightarrow{g} & E_2 \\ \varphi_2 \downarrow & & \downarrow \varphi_2 \\ \mathbb{C}^2 & \xrightarrow{G} & \mathbb{C}^2 \end{array}$$

commute. The fixed points of G are the solutions of the following system

$$\begin{cases} x &= x + \frac{1}{\lambda_0}y \\ y &= y \end{cases}$$

and this are $\{(x, 0) \in \mathbb{C}^2; x \in \mathbb{C}\}$. Then the points $\text{Fix}_1(g) = \{(x : 0 : 1) \in \mathbb{C}P(2); x \in \mathbb{C}\}$ are fixed by g . Now we consider the points of the form $(x : 1 : z) \in \mathbb{C}P(2)$. We have $g(x : 1 : z) = (\lambda_0 x + 1 : \lambda_0 : \lambda_0 z)$ and $G : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ defined by $G(x, z) = \left(x + \frac{1}{\lambda_0}, z\right)$ such that

$$\begin{array}{ccc} E_1 & \xrightarrow{g} & E_1 \\ \varphi_1 \downarrow & & \downarrow \varphi_1 \\ \mathbb{C}^2 & \xrightarrow{G} & \mathbb{C}^2 \end{array}$$

commute. Then fixed points of G are the solution of the system

$$\begin{cases} x &= x + \frac{1}{\lambda_0} \\ z &= z \end{cases}$$

Notice that this system doesn't have solutions. Therefore there are not fixed points of g of the form $(x : 1 : z) \in \mathbb{C}P(2)$. And now we consider the points of the form $(1 : y : z) \in \mathbb{C}P(2)$. We have

$g(1 : y : z) = (\lambda_0 + y : \lambda_0 y : \lambda_0 z)$ and $G : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ defined by $G(y, z) = \left(\frac{\lambda_0 y}{\lambda_0 + y}, \frac{\lambda_0 z}{\lambda_0 + y} \right)$ such that the following diagram is commutative

$$\begin{array}{ccc} E_0 & \xrightarrow{g} & E_0 \\ \varphi_0 \downarrow & & \downarrow \varphi_0 \\ \mathbb{C}^2 & \xrightarrow{G} & \mathbb{C}^2 \end{array}$$

if $y \neq -\lambda_0$. Notice that if $y = -\lambda_0$ then there are not fixed points of g . The fixed points of G are given by the solutions of the system

$$\begin{cases} y &= \frac{\lambda_0 y}{\lambda_0 + y} \\ z &= \frac{\lambda_0 z}{\lambda_0 + y} \end{cases}$$

and these are the points $\{(0, z) \in \mathbb{C}^2; z \in \mathbb{C}\}$. Therefore $\text{Fix}_3(g) = \{(1 : 0 : z) \in \mathbb{C}P(2); z \in \mathbb{C}\}$ is a subset of the fixed points of $g \in \text{Aut}(\mathbb{C}P(2))$. Therefore the fixed points of g in this case are two projective lines

$$\text{Fix}(g) = \{(x : 0 : 1) \in \mathbb{C}P(2); x \in \mathbb{C}\} \cup \{(1 : 0 : z) \in \mathbb{C}P(2); z \in \mathbb{C}\}.$$

Case (iii.3). We have

$$J = \begin{bmatrix} \lambda_0 & 1 & 0 \\ 0 & \lambda_0 & 1 \\ 0 & 0 & \lambda_0 \end{bmatrix},$$

and f is conjugate to $g(x : y : z) = (\lambda_0 x + y : \lambda_0 y + z : \lambda_0 z)$. Let us study the fixed points of g . First, we consider the points of the following form $(x : y : 1) \in \mathbb{C}P(2)$. Then we have $g(x : y : 1) = (\lambda_0 x + y : \lambda_0 y + 1 : \lambda_0)$ and $G : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ defined by $G(x, y) = \left(x + \frac{y}{\lambda_0}, y + \frac{1}{\lambda_0} \right)$ such that

$$\begin{array}{ccc} E_2 & \xrightarrow{g} & E_2 \\ \varphi_2 \downarrow & & \downarrow \varphi_2 \\ \mathbb{C}^2 & \xrightarrow{G} & \mathbb{C}^2 \end{array}$$

commute. The fixed points of G are given by the system

$$\begin{cases} x &= x + \frac{1}{\lambda_0} y \\ y &= y + \frac{1}{\lambda_0} \end{cases}$$

Therefore there are not fixed points of $g \in \text{Aut}(\mathbb{C}P(2))$ of the form $(x : y : 1) \in \mathbb{C}P(2)$ in this case, because there are not solutions for this system. Now we consider points of the form $(x : 1 : z) \in \mathbb{C}P(2)$. We obtain $g(x : 1 : z) = (\lambda_0 x + 1 : \lambda_0 + z : \lambda_0 z)$ and $G : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ defined by $G(x, z) = \left(\frac{\lambda_0 x + 1}{\lambda_0 + z}, \frac{\lambda_0 z}{\lambda_0 + z} \right)$ such that

$$\begin{array}{ccc} E_1 & \xrightarrow{g} & E_1 \\ \varphi_1 \downarrow & & \downarrow \varphi_1 \\ \mathbb{C}^2 & \xrightarrow{G} & \mathbb{C}^2 \end{array}$$

commute if $z \neq -\lambda_0$. Notice that if $z = -\lambda_0$ then there are not fixed points of $g \in \text{Aut}(\mathbb{C}P(2))$. Then the fixed points of g are given by the fixed points of G and these are given by the solutions of the system

$$\begin{cases} x &= \frac{\lambda_0 x + 1}{\lambda_0 z} \\ z &= \frac{\lambda_0 z}{\lambda_0 + z} \end{cases}$$

Therefore there are not fixed points of g in this case. By analogy with it we consider the points of the form $(1 : y : z) \in \mathbb{C}P(2)$. We obtain $g(1 : y : z) = (\lambda_0 + y : \lambda_0 y + z : \lambda_0 z)$ and $G : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ defined by $G(y, z) = \left(\frac{\lambda_0 y + z}{\lambda_0 + y}, \frac{\lambda_0 z}{\lambda_0 + y} \right)$ such that

$$\begin{array}{ccc} E_0 & \xrightarrow{g} & E_0 \\ \varphi_0 \downarrow & & \downarrow \varphi_0 \\ \mathbb{C}^2 & \xrightarrow{G} & \mathbb{C}^2 \end{array}$$

commute if $y \neq -\lambda_0$. Observe that if $y = -\lambda_0$ then there are not fixed points of g . The fixed points of G are given by the system

$$\begin{cases} y &= \frac{\lambda_0 y + z}{\lambda_0 + y} \\ z &= \frac{\lambda_0 z}{\lambda_0 + y} \end{cases}$$

And then $(0, 0)$ is the fixed point by G . Therefore $(1 : 0 : 0)$ is the only fixed point by g . And we have finished the proof of Theorem 4 \square

Now we obtain the following

Definition 1. We shall say that the biholomorphism $f : \mathbb{C}P(2) \rightarrow \mathbb{C}P(2)$ except for identity application is:

- (i) of type P1 if one point is fixed by f ;
- (ii) of type P2 if two points are fixed by f ;
- (iii) of type P3 if three points are fixed by f ;
- (iv) of type R2 if two projective lines are fixed by f ;
- (v) of type P1R2 if one point and two projective lines are fixed by f .

3 Construction of Riccati foliations on $\overline{\mathbb{C}} \times \mathbb{C}P(2)$

In this section we address the following question:

Question 1. *Let be given elements f_1, \dots, f_k of the group $\text{Aut}(\mathbb{C}P(2))$. Is there a Riccati foliation \mathcal{F} on $\overline{\mathbb{C}} \times \mathbb{C}P(2)$ such that the global holonomy of \mathcal{F} is conjugate to the subgroup of $\text{Aut}(\mathbb{C}P(2))$ generated by f_1, \dots, f_k ?*

We proceed similarly to [7], that is, we construct a singular holomorphic foliation \mathcal{F} on $\overline{\mathbb{C}} \times \mathbb{C}P(2)$ by gluing together local foliations transverse to almost every fiber and given in a neighborhood of the invariant fibers by suitable local models given in terms of the normal form of the corresponding holonomy map as in Theorem 4.

Proof of Theorem 1. Let $f_0 = (f_1 \circ \dots \circ f_k)^{-1}$ be a biholomorphism and let $x_0 = 0, x_1, \dots, x_k$ be points in \mathbb{C} . For each $j \in \{0, 1, \dots, k\}$ let D_j be a disk of radius $r > 0$ and center x_j such that $|x_i - x_j| > 2r$ for all $i \neq j$, $0 \leq i, j \leq k$. For each $j \in \{1, \dots, k\}$ we choose $x'_j = x_j + \frac{r}{2} \in D_j \setminus \{x_j\}$ and $x''_j = \frac{r}{2} \exp(\frac{2\pi\sqrt{-1}(j-1)}{k}) \in D_0 \setminus \{0\}$.

Let $\alpha_1, \dots, \alpha_k : [0, 1] \rightarrow \mathbb{C}$ be simple curves such that (i) $\alpha_j(0) = x''_j$ and $\alpha_j(1) = x'_j$; (ii) $\alpha_j([0, 1]) \cap D_i = \emptyset$ if $i \neq j$, $i \neq 0$; (iii) $\alpha_i([0, 1]) \cap \alpha_j([0, 1]) = \emptyset$ if $i \neq j$; (iv) for every $j \in \{1, \dots, k\}$, $\alpha_j([0, 1]) \cap D_0$ and $\alpha_j([0, 1]) \cap D_j$ are segments of straight lines contained in diameters of D_0 and D_j respectively.

Let A_1, \dots, A_k be tubular neighborhoods of $\alpha_1, \dots, \alpha_k$ respectively such that (v) $A_j \cap D_i = \emptyset$ if $i \neq j$, $i \neq 0$; (vi) $A_i \cap A_j = \emptyset$ if $i \neq j$; (vii) $A_j \cap D_0$ and $A_j \cap D_j$ are contained in sectors of D_0 and D_j , $1 \leq j \leq k$, respectively.

Let $U = (\bigcup_{j=1}^k A_j) \cup (\bigcup_{j=0}^k D_j)$ be a set and let $\gamma = \partial U$ be a simple curve. Let T be a tubular neighborhood of γ and let $V = (\overline{\mathbb{C}} \setminus U) \cup T$ be a set. Then $\{A_1, \dots, A_k, D_0, \dots, D_k, V\}$ is a covering of $\overline{\mathbb{C}}$ by open sets. For every $j \in \{1, \dots, k\}$ we consider affine coordinates (x, U_j, V_j) in $A_j \times \mathbb{C}^2 \hookrightarrow A_j \times \mathbb{C}P(2)$, $x \in A_j$, $(U_j, V_j) \in \mathbb{C}^2$. For each $i \in \{0, 1, \dots, k\}$ we consider affine coordinates (x, u_i, v_i) in $D_i \times \mathbb{C}^2 \hookrightarrow D_i \times \mathbb{C}P(2)$, $x \in D_i$, $(u_i, v_i) \in \mathbb{C}^2$. Put affine coordinates (w, y_1, y_2) in $V \times \mathbb{C}^2$ where $w = \frac{1}{x} \in V$ and $(y_1, y_2) \in \mathbb{C}^2$.

We take in each set of the form $A_j \times \mathbb{C}^2$, $V \times \mathbb{C}^2$ and $D_i \times \mathbb{C}^2$ a local model of foliation and glue them together. The local models are as follows:

1. In $A_j \times \mathbb{C}P(2)$ we consider the horizontal foliation whose leaves are of the form $A_j \times \{p\}$, $p \in \mathbb{C}P(2)$ for each $j \in \{1, \dots, k\}$.
2. In $V \times \mathbb{C}P(2)$ we consider the horizontal foliation whose leaves are of the form $V \times \{p\}$, $p \in \mathbb{C}P(2)$.
3. In $D_i \times \mathbb{C}^2$ we consider the singular holomorphic foliation \mathcal{F}_i induced by the vector field X_i in $D_i \times \mathbb{C}^2$ for every $i \in \{0, 1, \dots, k\}$. Put $l \in \{0, 1, \dots, k\}$. There exists an affine coordinate such that $f_l : E_0 \rightarrow E_0$ can be written in one of the following forms:

- (a) $f_l(u, v) = (u + \mu_l v, v + \mu_l)$ if f_l is of type P1.
- (b) $f_l(u, v) = (\mu_l u + \nu_l v, \mu_l v)$ if f_l is of type P2.
- (c) $f_l(u, v) = (\lambda'_l u, \lambda''_l v)$ if f_l is P3.
- (d) $f_l(u, v) = (\lambda''_l u, \lambda'_l v)$ if f_l is R2.
- (e) $f_l(u, v) = (u + \nu_l v, v)$ if f_l is of type P1R2.

where $\lambda'_l, \lambda''_l, \mu_l, \nu_l \in \mathbb{C} \setminus \{0\}$ are different.

• In case (c) ((d) respectively) we consider the singular holomorphic foliation \mathcal{F}_j on $D_j \times \mathbb{C}^2$ given by the vector field

$$X_j(x, u_j, v_j) = (x - x_j) \frac{\partial}{\partial x} + \alpha'_j u_j \frac{\partial}{\partial u_j} + \alpha''_j v_j \frac{\partial}{\partial v_j} \quad (1)$$

where $\exp(2\pi\sqrt{-1}\alpha'_j) = \lambda'_j$ and $\exp(2\pi\sqrt{-1}\alpha''_j) = \lambda''_j$. (In case (d) the foliation \mathcal{F}_j is given by $X_j(x, u_j, v_j) = (x - x_j) \frac{\partial}{\partial x} + \alpha''_j u_j \frac{\partial}{\partial u_j} + \alpha'_j v_j \frac{\partial}{\partial v_j}$ where α'_j and λ'_j are the same, respectively.)

Let $\gamma_j(\theta) = (r_j \exp(\sqrt{-1}\theta) + x_j, 0, 0)$, $0 \leq \theta \leq 2\pi$ be a curve where $0 < r_j < r$. Let $\Sigma_j = \{p_j\} \times \mathbb{C}^2$, $p_j \in \gamma_j([0, 2\pi])$.

Assertion 1. *The holonomy transformation of \mathcal{F}_j associated to Σ_j and γ_j is of the form $(u_j, v_j) \mapsto (\lambda'_j u_j, \lambda''_j v_j)$ where the foliation \mathcal{F}_j on $D_j \times \mathbb{C}^2$ is induced by equation 1.*

In fact, let $\Sigma_j = \{x_j + r_j\} \times \mathbb{C}^2$ be a local transverse section and let $p_j = (x_j + r_j, 0, 0) \in \Sigma_j$. Suppose $p_1 : D_j \times \mathbb{C}^2 \rightarrow D_j$, $p_1(x, y, z) = x$. Observe that the fibers $p_1^{-1}(x)$, $x \neq x_j$, are transverse to \mathcal{F} . Let $q = (x_j + r_j, u_j, v_j) \in \Sigma_j$ and let $\gamma_q(\theta) = (x(\theta), u_j(\theta), v_j(\theta))$ be the lifting of γ_j by p_1 with base point q . Therefore

$$x'(\theta) = p_1(\gamma'_q(\theta)) = p_1(\gamma'_j(\theta)) = \sqrt{-1}r_j \exp(\sqrt{-1}\theta),$$

and, if $Y_j = (u_j, v_j) \in \mathbb{C}^2$ then

$$\frac{Y'_j}{x'} = \frac{Y'_j}{\sqrt{-1}r_j \exp(\sqrt{-1}\theta)}.$$

On the other hand, by equation 1 we have

$$\frac{dx}{dT} = x - x_j$$

and

$$\frac{dY_j}{dT} = \begin{bmatrix} \alpha'_j & 0 \\ 0 & \alpha''_j \end{bmatrix} \cdot \begin{bmatrix} u_j \\ v_j \end{bmatrix} = AY_j$$

so we obtain

$$\frac{dY_j}{dx} = \frac{dY_j}{dT} \cdot \frac{dT}{dx} = \frac{\frac{dY_j}{dT}}{\frac{dx}{dT}} = \frac{AY_j}{x - x_j}$$

and we have

$$\frac{Y'_j}{\sqrt{-1}r_j \exp(\sqrt{-1}\theta)} = \frac{Y'_j}{x'} = \frac{AY_j}{r_j \exp(\sqrt{-1}\theta)}$$

then $Y'_j = \sqrt{-1}AY_j$. Notice that the solution of $Y'_j = \sqrt{-1}AY_j$ such that $Y_j(0) = (u_j, v_j)$ is $Y_j(\theta) = \exp(\sqrt{-1}\theta A) \cdot Y_j(0)$. Therefore the holonomy is the biholomorphism $f : \Sigma_j \rightarrow \Sigma_j$ defined

by

$$\begin{aligned}
f(u_j, v_j) &= Y_j(2\pi) \\
&= \exp\left(2\pi\sqrt{-1}\begin{bmatrix} \alpha'_j & 0 \\ 0 & \alpha''_j \end{bmatrix}\right) \cdot \begin{bmatrix} u_j \\ v_j \end{bmatrix} \\
&= \begin{bmatrix} \exp(2\pi\sqrt{-1}\alpha'_j) & 0 \\ 0 & \exp(2\pi\sqrt{-1}\alpha''_j) \end{bmatrix} \cdot \begin{bmatrix} u_j \\ v_j \end{bmatrix} \\
&= (\exp(2\pi\sqrt{-1}\alpha'_j)u_j, \exp(2\pi\sqrt{-1}\alpha''_j)v_j) \\
&= (\lambda'_j u_j, \lambda''_j v_j)
\end{aligned}$$

and this proves the assertion. (In case (d) we prove the holonomy transformation of \mathcal{F}_j associated to Σ_j and γ_j is $(u_j, v_j) \mapsto (\lambda'_j u_j, \lambda''_j v_j)$ respectively.)

• In case (e) we consider the foliation \mathcal{F}_j on $D_j \times \mathbb{C}^2$ given by

$$X_j(x, u_j, v_j) = (x - x_j) \frac{\partial}{\partial x} + \frac{\nu_j}{2\pi\sqrt{-1}} v_j \frac{\partial}{\partial u_j}. \quad (2)$$

Let $\gamma_j(\theta) = (r_j \exp(\sqrt{-1}\theta) + x_j, 0, 0)$, $0 \leq \theta \leq 2\pi$ be a curve where $0 < r_j < r$. Let $\Sigma_j = \{p_j\} \times \mathbb{C}^2$, $p_j \in \gamma_j([0, 2\pi])$.

Assertion 2. *The holonomy transformation of \mathcal{F}_j associated to Σ_j and γ_j is the following form $(u_j, v_j) \mapsto (u_j + \nu_j v_j, v_j)$, where the foliation \mathcal{F}_j on $D_j \times \mathbb{C}^2$ is given by equation 2.*

In fact, let $\Sigma_j = \{x_j + r_j\} \times \mathbb{C}^2$ be a local transverse section and let $p_j = (x_j + r_j, 0, 0) \in \Sigma_j$. Suppose $p_1 : D_j \times \mathbb{C}^2 \rightarrow D_j$, $p_1(x, y, z) = x$. Notice that the fibers $p_1^{-1}(x)$, $x \neq x_j$ are transverse to \mathcal{F} . Let $q = (x_j + r_j, u_j, v_j) \in \Sigma_j$ and let $\gamma_q(\theta) = (x(\theta), u_j(\theta), v_j(\theta))$ be the lifting of γ_j by p_1 with base point q . Therefore $x'(\theta) = p_1(\gamma'_q(\theta)) = p_1(\gamma'_j(\theta)) = \sqrt{-1}r_j \exp(\sqrt{-1}\theta)$, and, if $Y_j = (u_j, v_j) \in \mathbb{C}^2$ then

$$\frac{Y'_j}{x'} = \frac{Y'_j}{\sqrt{-1}r_j \exp(\sqrt{-1}\theta)}.$$

On the other hand, by equation 2 we have $\frac{dx}{dT} = x - x_j$ and

$$\frac{dY_j}{dT} = \begin{bmatrix} 0 & \frac{\nu_j}{2\pi\sqrt{-1}} \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} u_j \\ v_j \end{bmatrix} = AY_j,$$

so we obtain

$$\frac{dY_j}{dx} = \frac{dY_j}{dT} \cdot \frac{dT}{dx} = \frac{\frac{dY_j}{dT}}{\frac{dx}{dT}} = \frac{AY_j}{x - x_j}$$

and we have got

$$\frac{Y'_j}{\sqrt{-1}r_j \exp(\sqrt{-1}\theta)} = \frac{Y'_j}{x'} = \frac{AY_j}{r_j \exp(\sqrt{-1}\theta)}$$

therefore $Y'_j = \sqrt{-1}AY_j$. Observe that the solution of $Y'_j = \sqrt{-1}AY_j$ with $Y_j(0) = (u_j, v_j)$ is $Y_j(\theta) = \exp(\sqrt{-1}\theta A) \cdot Y_j(0)$. Therefore the holonomy is the biholomorphism $f : \Sigma_j \rightarrow \Sigma_j$ defined by

$$\begin{aligned}
f(u_j, v_j) &= Y_j(2\pi) \\
&= \exp\left(2\pi\sqrt{-1}\begin{bmatrix} 0 & \frac{\nu_j}{2\pi\sqrt{-1}} \\ 0 & 0 \end{bmatrix}\right) \cdot \begin{bmatrix} u_j \\ v_j \end{bmatrix} \\
&= \exp\left(\begin{bmatrix} 0 & \nu_j \\ 0 & 0 \end{bmatrix}\right) \cdot \begin{bmatrix} u_j \\ v_j \end{bmatrix} \\
&= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & \nu_j \\ 0 & 0 \end{bmatrix}\right) \cdot \begin{bmatrix} u_j \\ v_j \end{bmatrix} \\
&= \begin{bmatrix} 1 & \nu_j \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_j \\ v_j \end{bmatrix} \\
&= (u_j + \nu_j v_j, v_j)
\end{aligned}$$

and it proves the assertion.

In case (b) we consider the singular foliation \mathcal{F} on $D_j \times \mathbb{C}^2$ given by

$$X_j(x, u_j, v_j) = (x - x_j)\frac{\partial}{\partial x} + (\lambda u_j + \frac{\nu}{2\pi\sqrt{-1}\mu}v_j)\frac{\partial}{\partial u_j} + \lambda v_j\frac{\partial}{\partial v_j}. \quad (3)$$

where $\exp(2\pi\sqrt{-1}\lambda) = \mu$.

Let $\gamma_j(\theta) = (r_j \exp(\sqrt{-1}\theta) + x_j, 0, 0)$, $0 \leq \theta \leq 2\pi$ be a curve where $0 < r_j < r$. Let $\Sigma_j = \{p_j\} \times \mathbb{C}^2$, $p_j \in \gamma_j([0, 2\pi])$.

Assertion 3. *The holonomy transformation of \mathcal{F}_j associated to Σ_j and γ_j is of the form $(u_j, v_j) \mapsto (\mu u_j + \nu v_j, \mu v_j)$ where the foliation \mathcal{F}_j on $D_j \times \mathbb{C}^2$ is induced by equation 3.*

In fact, let $\Sigma_j = \{x_j + r_j\} \times \mathbb{C}^2$ be a local transverse section and let $p_j = (x_j + r_j, 0, 0) \in \Sigma_j$. Suppose $p_1 : D_j \times \mathbb{C}^2 \rightarrow D_j$, $p_1(x, y, z) = x$. Notice that the fibers $p_1^{-1}(x)$, $x \neq x_j$ are transverse to \mathcal{F} . Let $q = (x_j + r_j, u_j, v_j) \in \Sigma_j$ and let $\gamma_q(\theta) = (x(\theta), u_j(\theta), v_j(\theta))$ be the lifting of γ_j by p_1 with base point q . Therefore

$$x'(\theta) = p_1(\gamma'_q(\theta)) = p_1(\gamma'_j(\theta)) = \sqrt{-1}r_j \exp(\sqrt{-1}\theta),$$

and, if $Y_j = (u_j, v_j) \in \mathbb{C}^2$ then $\frac{Y'_j}{x'} = \frac{Y'_j}{\sqrt{-1}r_j \exp(\sqrt{-1}\theta)}$. On the other hand, by equation 3 we have $\frac{dx}{dT} = x - x_j$ and

$$\frac{dY_j}{dT} = \begin{bmatrix} \lambda & \frac{\nu}{2\pi\sqrt{-1}\mu} \\ 0 & \lambda \end{bmatrix} \cdot \begin{bmatrix} u_j \\ v_j \end{bmatrix} = AY_j,$$

so we obtain

$$\frac{dY_j}{dx} = \frac{dY_j}{dT} \cdot \frac{dT}{dx} = \frac{\frac{dY_j}{dT}}{\frac{dx}{dT}} = \frac{AY_j}{x - x_j}$$

and we have

$$\frac{Y'_j}{\sqrt{-1}r_j \exp(\sqrt{-1}\theta)} = \frac{Y'_j}{x'} = \frac{AY_j}{r_j \exp(\sqrt{-1}\theta)}$$

therefore $Y'_j = \sqrt{-1}AY_j$. Observe that the solution of $Y'_j = \sqrt{-1}AY_j$ such that $Y_j(0) = (u_j, v_j)$ is $Y_j(\theta) = \exp(\sqrt{-1}\theta A) \cdot Y_j(0)$. Therefore the holonomy is the biholomorphism $f : \Sigma_j \rightarrow \Sigma_j$ defined by

$$\begin{aligned} f(u_j, v_j) &= Y_j(2\pi) \\ &= \exp\left(2\pi\sqrt{-1}\begin{bmatrix} \lambda & \frac{\nu}{2\pi\sqrt{-1}\mu} \\ 0 & \lambda \end{bmatrix}\right) \cdot \begin{bmatrix} u_j \\ v_j \end{bmatrix} \\ &= \exp\left(\begin{bmatrix} 2\pi\sqrt{-1}\lambda & 0 \\ 0 & 2\pi\sqrt{-1}\lambda \end{bmatrix} + \begin{bmatrix} 0 & \frac{\nu}{\mu} \\ 0 & 0 \end{bmatrix}\right) \cdot \begin{bmatrix} u_j \\ v_j \end{bmatrix} \\ &= \exp\left(\begin{bmatrix} 2\pi\sqrt{-1}\lambda & 0 \\ 0 & 2\pi\sqrt{-1}\lambda \end{bmatrix}\right) \cdot \exp\left(\begin{bmatrix} 0 & \frac{\nu}{\mu} \\ 0 & 0 \end{bmatrix}\right) \begin{bmatrix} u_j \\ v_j \end{bmatrix} \\ &= \begin{bmatrix} \mu & 0 \\ 0 & \mu \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{\nu}{\mu} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) \begin{bmatrix} u_j \\ v_j \end{bmatrix} \\ &= \begin{bmatrix} \mu & 0 \\ 0 & \mu \end{bmatrix} \begin{bmatrix} 1 & \frac{\nu}{\mu} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_j \\ v_j \end{bmatrix} \\ &= \begin{bmatrix} \mu & \nu \\ 0 & \mu \end{bmatrix} \cdot \begin{bmatrix} u_j \\ v_j \end{bmatrix} \\ &= (\mu u_j + \nu v_j, \mu v_j) \end{aligned}$$

and this proves the assertion.

In case (a) we consider the foliation \mathcal{F}_j on $D_j \times \mathbb{C}^2$ given by

$$X_j(x, u_j, v_j) = (x - x_j) \frac{\partial}{\partial x} + \left(\frac{\mu}{2\pi\sqrt{-1}}v_j - \frac{\mu^2}{4\pi\sqrt{-1}}\right) \frac{\partial}{\partial u_j} + \frac{\mu}{2\pi\sqrt{-1}} \frac{\partial}{\partial v_j}. \quad (4)$$

Let $\gamma_j(\theta) = (r_j \exp(\sqrt{-1}\theta) + x_j, 0, 0)$, $0 \leq \theta \leq 2\pi$ be a curve where $0 < r_j < r$. Let $\Sigma_j = \{p_j\} \times \mathbb{C}^2$, $p_j \in \gamma_j([0, 2\pi])$.

Assertion 4. *The holonomy transformation of \mathcal{F}_j associated to Σ_j and γ_j is of the form $(u_j, v_j) \mapsto (u_j + \mu v_j, v_j + \mu)$.*

In fact, let $\Sigma_j = \{x_j + r_j\} \times \mathbb{C}^2$ be a local transverse section and let $p_j = (x_j + r_j, 0, 0) \in \Sigma_j$. Suppose $p_1 : D_j \times \mathbb{C}^2 \rightarrow D_j$, $p_1(x, y, z) = x$. Notice that the fibers $p_1^{-1}(x)$, $x \neq x_j$ are transverse to \mathcal{F} . Let $q = (x_j + r_j, u_j, v_j) \in \Sigma_j$ and let $\gamma_q(\theta) = (x(\theta), u_j(\theta), v_j(\theta))$ be the lifting of γ_j by p_1 with base point q . Therefore

$$x'(\theta) = p_1(\gamma'_q(\theta)) = p_1(\gamma'_j(\theta)) = \sqrt{-1}r_j \exp(\sqrt{-1}\theta),$$

and, if $Y_j = (u_j, v_j) \in \mathbb{C}^2$ then

$$\frac{Y'_j}{x'} = \frac{Y'_j}{\sqrt{-1}r_j \exp(\sqrt{-1}\theta)}.$$

On the other hand, by equation 2 we have

$$\frac{dx}{dT} = x - x_j$$

and

$$\frac{dY_j}{dT} = \begin{bmatrix} 0 & \frac{\mu}{2\pi\sqrt{-1}} \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} u_j \\ v_j \end{bmatrix} + \begin{bmatrix} -\frac{\mu^2}{4\pi\sqrt{-1}} \\ \frac{\mu}{2\pi\sqrt{-1}} \end{bmatrix} = AY_j + B,$$

so we obtain $\frac{dY_j}{dx} = \frac{dY_j}{dT} \cdot \frac{dT}{dx} = \frac{\frac{dY_j}{dT}}{\frac{dx}{dT}} = \frac{AY_j + B}{x - x_j}$ and we have

$$\frac{Y'_j}{\sqrt{-1}r_j \exp(\sqrt{-1}\theta)} = \frac{Y'_j}{x'} = \frac{AY_j + B}{r_j \exp(\sqrt{-1}\theta)}$$

therefore $Y'_j = \sqrt{-1}AY_j + \sqrt{-1}B$. Observe that the solution of

$$Y'_j = \sqrt{-1}AY_j + \sqrt{-1}B$$

with $Y_j(0) = (u_j, v_j)$ is

$$Y_j(\theta) = \exp(\sqrt{-1}\theta A) \cdot \left[\int_0^\theta \exp(\sqrt{-1}s A) \cdot B(s) ds + Y_j(0) \right],$$

that is,

$$Y_j(\theta) = \left(-\frac{\mu^2}{4\pi}\theta - \frac{1}{2}\frac{\mu^2}{4\pi^2}\theta^2 + u_j + \frac{\mu^2}{4\pi^2}\theta^2 + \frac{\mu}{2\pi}v_j\theta, \frac{\mu}{2\pi}\theta + v_j \right)$$

Therefore the holonomy is the biholomorphism $f : \Sigma_j \rightarrow \Sigma_j$ defined by

$$f(u_j, v_j) = Y_j(2\pi) = (u_j + \mu v_j, v_j + \mu)$$

and it proves the assertion.

Let us glue together the foliation on $A_j \times \mathbb{C}P(2)$ and the foliations on $D_j \times \mathbb{C}P(2)$. First we consider f_j of type P3 or R2. Then we are in case (c) or (d). Observe that $A_j \cap D_j$ is simply connected and $x_j \notin A_j \cap D_j$, we consider the coordinate system $(x, \tilde{u}_j, \tilde{v}_j)$ in $(A_j \cap D_j) \times \mathbb{C}P(2)$ such that

$$\tilde{u}_j = u_j \exp(-\alpha'_j \log(\frac{x - x_j}{\frac{r}{2}}))$$

and

$$\tilde{v}_j = v_j \exp(-\alpha''_j \log(\frac{x - x_j}{\frac{r}{2}}))$$

where \log is the branch of the logarithm in $\overline{\mathbb{C}} \setminus \{x + \sqrt{-1}y; x \leq 0\}$ such that $\log(1) = 0$. Observe that $x'_j = x_j + \frac{r}{2}$ implies that

$$\tilde{u}_j(x'_j, u_j) = u_j \exp(-\alpha'_j \log(1)) = u_j, \quad \tilde{v}_j(x'_j, v_j) = v_j$$

and $\tilde{u}_j(x, 0) = 0$, $\tilde{v}_j(x, 0) = 0$. Therefore the leaves of the foliation on $(A_j \cap D_j) \times \mathbb{C} P(2)$ are given by $(\tilde{u}_j, \tilde{v}_j) = \text{constant}$. Let us identify the point $(x, U_j, V_j) \in (A_j \cap D_j) \times \mathbb{C}^2 \subset A_j \times \mathbb{C}^2$ with the point $((x, u_j, v_j)) \in (A_j \cap D_j) \times \mathbb{C}^2 \subset D_j \times \mathbb{C}^2$, where

$$u_j = U_j \exp(\alpha'_j \log(\frac{x - x_j}{\frac{r}{2}})) \quad (5)$$

and

$$v_j = V_j \exp(\alpha''_j \log(\frac{x - x_j}{\frac{r}{2}})). \quad (6)$$

Notice that with equation 5 and equation 6 we are gluing together in $(A_j \cap D_j) \times \mathbb{C}^2$ plaques of the foliation $\tilde{\mathcal{F}}$ on $A_j \times \mathbb{C}^2$ with plaques of foliation $\hat{\mathcal{F}}$ on $D_j \times \mathbb{C}^2$. Observe that this identification sends the fiber $\{x = c\} \subset A_j \times \mathbb{C}^2$, $c \in A_j \cap D_j$, in the fiber $\{x = c\} \subset D_j \times \mathbb{C}^2$, and the holonomy of the curve $\beta_j = \alpha_j * \gamma_j * \alpha_j^{-1}$ in the section $\Sigma''_j = \{x''_j\} \times \mathbb{C}^2 \subset A_j \times \mathbb{C}^2$ with respect to the foliation obtained by gluing together the $\tilde{\mathcal{F}}$ and $\hat{\mathcal{F}}$ is $(U_j, V_j) \mapsto (\lambda'_j U_j, \lambda''_j V_j)$.

If f_j is of type P1R2 we are in case (e) and the identifications 5 and 6 are

$$u_j = U_j - \frac{\nu_j}{2\pi\sqrt{-1}} V_j \log\left(\frac{x - x_j}{\frac{r}{2}}\right) \quad (7)$$

$$v_j = V_j \quad (8)$$

By analogy with it, if f_j is of type P2 we are in case (b) and the identifications are

$$u_j = \frac{1}{\lambda} \left((U_j - \frac{\nu}{2\pi\sqrt{-1}\mu} V_j) \exp(\lambda \log(\frac{x - x_j}{\frac{r}{2}})) \right) \quad (9)$$

$$v_j = V_j \exp(\lambda \log(\frac{x - x_j}{\frac{r}{2}})) \quad (10)$$

And if f_j is of type P1 we are in case (a) and the identifications are

$$u_j = U_j + \left(\frac{\mu}{2\pi\sqrt{-1}} V_j + \frac{\mu^2}{(2\pi\sqrt{-1})^2} \log(\frac{x - x_j}{\frac{r}{2}}) + \frac{\mu^2}{4\pi\sqrt{-1}} \right) \cdot \log(\frac{x - x_j}{\frac{r}{2}}) \quad (11)$$

$$v_j = V_j + \frac{\mu}{2\pi\sqrt{-1}} \log(\frac{x - x_j}{\frac{r}{2}}) \quad (12)$$

Now let us glue together the new foliation on $(A_j \cup D_j) \times \mathbb{C}^2$ with the foliation on $(A_j \cap D_0) \times \mathbb{C}^2$ identify the points $(x, U_j, V_j) \in ((A_j \cup D_j) \cap D_0) \times \mathbb{C}^2$ with $(x, u_0, v_0) \in (A_j \cap D_0) \times \mathbb{C}^2 \subset D_0 \times \mathbb{C}^2$ by

$$u_0 = U_j \exp(\alpha'_0 \log(\frac{x}{x''_j})) \quad (13)$$

and

$$v_0 = V_j \exp(\alpha''_0 \log(\frac{x}{x''_j})), \quad (14)$$

where $\lambda'_0 = \exp(2\pi\sqrt{-1}\alpha'_0)$ and $\lambda''_0 = \exp(2\pi\sqrt{-1}\alpha''_0)$, if f_0 is P3 or R2.

Notice that equation 13 e equation 14 glue together plaques of the foliation on $(A_j \cup D_j) \times \mathbb{C}^2$ with plaques of the foliation on $(A_j \cap D_0) \times \mathbb{C}^2$ and this defines a new foliation \mathcal{F} such that $D_0 \cup A_j \cup D_j$ is a leaf. The holonomy of the curve β_j in the section $\Sigma_j'' = \{x_j''\} \times \mathbb{C}^2 \subset D_0 \times \mathbb{C}^2$ is given by $(U_0, V_0) \mapsto (\lambda_j' U_0, \lambda_j'' V_0)$. Now suppose $\gamma_0(\theta) = \frac{r}{2} \exp(\sqrt{-1}\theta)$, $0 \leq \theta \leq 2\pi$, and for every $j = 1, \dots, k$ let μ_j be the segment of γ_0 between x_j'' and $\frac{r}{2}$ in the positive sense. Let

$$\delta_j = \mu_j * \beta_j * \mu_j^{-1} = \mu_j * \alpha_j * \gamma_j * \alpha_j^{-1} * \mu_j^{-1},$$

where $\gamma_j(\theta) = \frac{r}{2} \exp(\sqrt{-1}\theta) + x_j$, $\theta \in [0, 2\pi]$ and $\Sigma_0 = \{\frac{r}{2}\} \times \mathbb{C}^2$. The holonomy of the curve δ_j in Σ_0 is $(U, V) \mapsto f_j(U, V)$.

By analogy with this case, we have the identifications

$$\begin{aligned} u_0 &= U_j - \frac{\nu_j}{2\pi\sqrt{-1}} V_j \log\left(\frac{x}{x_j''}\right) \\ v_0 &= V_j \end{aligned}$$

if f_0 is of type P1R2,

$$\begin{aligned} u_0 &= \frac{1}{\lambda} \left((U_j - \frac{\nu}{2\pi\sqrt{-1}\mu} V_j) \exp(\lambda \log(\frac{x - x_j}{\frac{r}{2}})) \right) \\ v_0 &= V_j \exp(\lambda \log(\frac{x}{x_j''})) \end{aligned}$$

if f_0 is of type P2 and

$$\begin{aligned} u_0 &= U_j + \left(\frac{\mu}{2\pi\sqrt{-1}} V_j + \frac{\mu^2}{(2\pi\sqrt{-1})^2} \log(\frac{x}{x_j''}) + \frac{\mu^2}{4\pi\sqrt{-1}} \cdot \log(\frac{x}{x_j''}) \right) \\ v_0 &= V_j + \frac{\mu}{2\pi\sqrt{-1}} \log(\frac{x}{x_j''}) \end{aligned}$$

if f_j is of type P1 respectively.

Now, let $\widetilde{M} = \overline{\mathbb{C}} \times \mathbb{C}^2$ be a complex manifold and let $\widetilde{\mathcal{F}}$ be a foliation obtained at the end of the process. By construction the holonomy of the leaf $U = \left(\bigcup_{i=0}^k A_i\right) \cap \left(\bigcup_{j=0}^k D_j\right)$ in Σ_0 is generated by f_1, \dots, f_k and the holonomy of the curve $\delta_1 * \dots * \delta_k * \gamma_0$ is the identity. Notice that \widetilde{M} admits the vertical foliation $x = \text{constant}$ on $A_j \times \mathbb{C}^2$, $D_j \times \mathbb{C}^2$, $D_0 \times \mathbb{C}^2$ and it cuts U at a single point and so we can define a projection $\widetilde{p}: \widetilde{M} \rightarrow U$ such that $\widetilde{p}^{-1}(x)$ is the leaf of this new foliation. Observe that this new foliation is transverse to $\widetilde{\mathcal{F}}$ in $\widetilde{M} \setminus \bigcup_{j=0}^k \{x = x_j\}$. Suppose the annulus $A = T \cap U$, if δ is a closed curve in A which generates the homotopy of A , then the holonomy of δ with respect to $\widetilde{\mathcal{F}}$ in some transversal section is trivial, because δ is homotopic to the curve $\delta_1 * \dots * \delta_k * \gamma_0$ in $U \setminus \bigcup_{j=0}^k \{x_j\}$ and the holonomy of this is trivial. Then we use the holonomy and obtain that the restricted foliation $\widetilde{\mathcal{F}}|_{\widetilde{p}^{-1}(A)}$ is a product foliation, that is, there is a biholomorphism φ of the some neighborhood W of $A \subset \widetilde{p}^{-1}(A)$ onto $A \times \Delta$, where $\Delta \subset \mathbb{C}^2$ is a polydisc such that it sends leaves of $\widetilde{\mathcal{F}}|_W$ onto leaves $A \times \{c\}$, $c \in \Delta$ of the trivial foliation.

Now we glue together the foliations $\widetilde{\mathcal{F}}$ in \widetilde{M} and $\widehat{\mathcal{F}}$ in $V \times D$ by φ .

Observe that we use the same ideas in the local model of foliation in other affine coordinates of $\mathbb{C}P(2)$ and it is proves the result. \square

4 Normal forms of Riccati foliations

Now we prove Theorems 2 and 3.

Proof of Theorem 2. We consider a singular holomorphic foliation \mathcal{F} on $\overline{\mathbb{C}} \times \overline{\mathbb{C}}^n$ given by a polynomial vector field X in affine coordinates $(x, y) \in \mathbb{C} \times \mathbb{C}^n \hookrightarrow \overline{\mathbb{C}} \times \overline{\mathbb{C}}^n$ and assume that \mathcal{F} is transverse to almost every fiber of η . Write $X(x, y) = P(x, y) \frac{\partial}{\partial x} + Q_1(x, y) \frac{\partial}{\partial y_1} + \cdots + Q_n(x, y) \frac{\partial}{\partial y_n}$. If $\{x'_0\} \times \overline{\mathbb{C}}^n$ is not an invariant fiber by \mathcal{F} , then \mathcal{F} by compactness \mathcal{F} is transverse to $\{x\} \times \overline{\mathbb{C}}^n$, $\forall x \approx x'_0$. Hence $P(x, y) \neq 0$, $\forall x \approx x'_0$, $\forall y \in \mathbb{C}^n$ and thus $P(x, y) = p(x)$.

Claim 1. We have $\deg_{y_n}(Q_n) \leq 2$ (where $\deg_{y_n}(\cdot)$ denotes the degree with respect to the variable y_n).

Proof. Suppose $\deg_{y_n}(Q_n) > 2$ and write $\deg_{y_n}(Q_n) = m + 2$ for some $m \in \mathbb{N}$. The foliation \mathcal{F} is given by the meromorphic vector field

$$X = p \frac{\partial}{\partial x} + Q_1 \frac{\partial}{\partial y_1} + \cdots + Q_{n-1} \frac{\partial}{\partial y_{n-1}} - w_n^2 \frac{1}{w_n^{m+2}} \tilde{Q}_n \frac{\partial}{\partial w_n}$$

in affine coordinates $(x, y_1, \dots, y_{n-1}, w_n) \in \mathbb{C} \times \mathbb{C}^n$ where $w_n = \frac{1}{y_n}$ and \tilde{Q}_n is a polynomial in $\mathbb{C}[x, y_1, \dots, y_{n-1}, w_n]$. If we multiply it by w_n^m , we obtain a polynomial vector field as

$$\tilde{X} = w_n^m p \frac{\partial}{\partial x} + w_n^m Q_1 \frac{\partial}{\partial y_1} + \cdots + w_n^m Q_{n-1} \frac{\partial}{\partial y_{n-1}} + \tilde{Q}_n \frac{\partial}{\partial w_n}$$

and we obtain three possibilities:

1. if $\deg_{w_n}(Q_1) = m$, then

$$\tilde{X} = w_n^m p \frac{\partial}{\partial x} + \tilde{Q}_1 \frac{\partial}{\partial y_1} + \cdots + w_n^m Q_{n-1} \frac{\partial}{\partial y_{n-1}} + \tilde{Q}_n \frac{\partial}{\partial w_n},$$

2. if $\deg_{w_n}(Q_1) > m$, that is, $\exists l \in \mathbb{N}$ such that $\deg_{w_n}(Q_1) = m + l$, then

$$\tilde{X} = w_n^m p \frac{\partial}{\partial x} + w_n^m \frac{1}{w_n^{m+l}} \tilde{Q}_1 \frac{\partial}{\partial y_1} + \cdots + w_n^m Q_{n-1} \frac{\partial}{\partial y_{n-1}} + \tilde{Q}_n \frac{\partial}{\partial w_n}.$$

We multiply it by w_n^l and obtain a new polynomial vector field

$$\tilde{\tilde{X}} = w_n^{m+l} p \frac{\partial}{\partial x} + \tilde{Q}_1 \frac{\partial}{\partial y_1} + \cdots + w_n^{m+l} Q_{n-1} \frac{\partial}{\partial y_{n-1}} + w_n^l \tilde{Q}_n \frac{\partial}{\partial w_n},$$

3. If $\deg_{w_n}(Q_1) < m$, that is, $\exists l \in \mathbb{N}$ such that $\deg_{w_n}(Q_1) = m - l$, then

$$\tilde{X} = w_n^m p \frac{\partial}{\partial x} + w_n^l \tilde{Q}_1 \frac{\partial}{\partial y_1} + \cdots + w_n^m Q_{n-1} \frac{\partial}{\partial y_{n-1}} + \tilde{Q}_n \frac{\partial}{\partial w_n}$$

where \tilde{Q}_1 is a polynomial in $\mathbb{C}[x, y_1, \dots, y_{n-1}, w_n]$. In all these cases we use the same ideas for all Q_j with $j = 2, \dots, n-1$ and it implies a polynomial vector field X without poles in affine coordinates $(x, y_1, \dots, y_{n-1}, w_n) \in \mathbb{C} \times \mathbb{C}^n \hookrightarrow \overline{\mathbb{C}} \times \overline{\mathbb{C}}^n$ such that \mathcal{F} is tangent to the fiber $\{x_0\} \times \overline{\mathbb{C}}^n$ in the set $\{x = x_0, w_n = 0\}$, and so it is a contradiction. \square

Therefore

$$X = p \frac{\partial}{\partial x} + Q_1 \frac{\partial}{\partial y_1} + \cdots + Q_{n-1} \frac{\partial}{\partial y_{n-1}} + q \frac{\partial}{\partial y_n}$$

where $q = q_{n,2}y_n^2 + q_{n,1}y_n + q_{n,0}$ with $q_{n,2}, q_{n,1}, q_{n,0} \in \mathbb{C}[x, y_1, \dots, y_{n-1}]$.

Now observe that $\deg_{y_n}(Q_j) = 0$ for all $j = 1, \dots, n-1$. In fact, assume that $\deg_{y_n}(Q_j) > 0$ so there exists $l_j \in \mathbb{N}$ such that $\deg_{y_n}(Q_j) = l_j$, then

$$X = p \frac{\partial}{\partial x} + Q_1 \frac{\partial}{\partial y_1} + \cdots + Q_{n-1} \frac{\partial}{\partial y_{n-1}} - \tilde{q} \frac{\partial}{\partial w_n}$$

where $\tilde{q} = q_{n,2} + q_{n,1}w_n + q_{n,0}w_n^2$ with $q_{n,2}, q_{n,1}$ and $q_{n,0}$ are polynomials in $\mathbb{C}[x, y_1, \dots, y_{n-1}]$. Let $m = \max\{\deg_{y_n}(Q_j); j = 1, \dots, n-1\}$. Suppose $m = \deg_{y_n}(Q_1)$. We multiply the vector field X by w_n^m so we obtain the vector field

$$X = w_n^m p \frac{\partial}{\partial x} + \tilde{Q}_1 \frac{\partial}{\partial y_1} + w_n^{\alpha_2} \tilde{Q}_2 \frac{\partial}{\partial y_2} + \cdots + w_n^{\alpha_{n-1}} \tilde{Q}_{n-1} \frac{\partial}{\partial y_{n-1}} - w_n^m \tilde{q} \frac{\partial}{\partial w_n}$$

with $\alpha_2 = m - \deg_{y_n}(Q_2)$ and $\alpha_{n-1} = m - \deg_{y_n}(Q_{n-1})$, where \tilde{Q}_j are polynomials in $\mathbb{C}[x, y_1, \dots, y_{n-1}, w_n]$ and $q_{n,k}$ are polynomials in $\mathbb{C}[x, y_1, \dots, y_{n-1}]$. We obtain \mathcal{F} is tangent to $\{x_0\} \times \overline{\mathbb{C}}^n$ in the set $\{x = x_0, w_n = 0\}$ so we have a contradiction. We obtain

$$X = p \frac{\partial}{\partial x} + Q_1 \frac{\partial}{\partial y_1} + \cdots + Q_{n-1} \frac{\partial}{\partial y_{n-1}} + q \frac{\partial}{\partial y_n}$$

where $q = q_{n,2}y_n^2 + q_{n,1}y_n + q_{n,0}$ with $q_{n,2}, q_{n,1}, q_{n,0}, Q_1, \dots, Q_{n-1}$ polynomials in $\mathbb{C}[x, y_1, \dots, y_{n-1}]$ and p is polynomial in $\mathbb{C}[x]$. By analogy with it we use the affine coordinates $(x, y_1, \dots, w_k, \dots, y_{n-1}, y_n) \in \mathbb{C}^{n+1} \hookrightarrow \overline{\mathbb{C}} \times \overline{\mathbb{C}}^n$, with $w_k = \frac{1}{y_k}$, $k \in \{1, \dots, n-1\}$ and we obtain $\deg_{y_k}(Q_k) \leq 2$, $\deg_{y_k}(Q_j) = 0$ for every $j \in \{1, \dots, n\} \setminus \{k\}$. Therefore

$$X = p \frac{\partial}{\partial x} + Q_1 \frac{\partial}{\partial y_1} + \cdots + Q_n \frac{\partial}{\partial y_n}$$

in affine coordinates $(x, y) \in \mathbb{C} \times \mathbb{C}^n \hookrightarrow \overline{\mathbb{C}} \times \overline{\mathbb{C}}^n$, $y = (y_1, \dots, y_n)$, where $Q_j = q_{j,2}(x)y_j^2 + q_{j,1}(x)y_j + q_{j,0}(x)$ with $q_{j,2}, q_{j,1}, q_{j,0}, p$ polynomials in $\mathbb{C}[x]$. And we prove Theorem 2. \square

Corollary 1. *Let \mathcal{F} be a singular holomorphic foliation on $\overline{\mathbb{C}} \times \overline{\mathbb{C}}^n$ given by a polynomial vector field X in affine coordinates on $\mathbb{C} \times \mathbb{C}^n$. Suppose \mathcal{F} is transverse at least one fiber $\{x_0\} \times \overline{\mathbb{C}}^n$ of η . Then \mathcal{F} is a Riccati foliation on $\overline{\mathbb{C}} \times \overline{\mathbb{C}}^n$.*

Proof. We consider

$$X(x, y) = P(x, y) \frac{\partial}{\partial x} + Q_1(x, y) \frac{\partial}{\partial y_1} + \cdots + Q_n(x, y) \frac{\partial}{\partial y_n}$$

in affine coordinates $(x, y) \in \mathbb{C} \times \mathbb{C}^n \hookrightarrow \overline{\mathbb{C}} \times \overline{\mathbb{C}}^n$, $y = (y_1, \dots, y_n)$ where P, Q_j are polynomials in $\mathbb{C}[x, y]$. If \mathcal{F} is transverse to $\{x_0\} \times \overline{\mathbb{C}}^n$, then $P(x, y) = p(x)$, $\forall x \in \mathbb{C}$. We use the transversality and it implies that there exists $\varepsilon > 0$ such that \mathcal{F} is transverse to $\{x_0\} \times \overline{\mathbb{C}}^n$, $\forall x \in D_\varepsilon(x_0)$. Now we use the Theorem 2. It implies that $\deg_{y_n}(Q_n) \leq 2$ and $\deg_{y_n}(Q_j) = 0$, $j \in \{1, \dots, n-1\}$. Then $\frac{\partial^k Q_j}{\partial y_n^k} = 0$ and $\frac{\partial Q_n}{\partial y_n} = 0$, $\forall k \geq 3$, $\forall x \in D_\varepsilon(x_0)$. Now we use the identity theorem [6] and obtain $\frac{\partial^k Q_j}{\partial y_n^k} = 0$ and $\frac{\partial Q_n}{\partial y_n} = 0$ in $\overline{\mathbb{C}} \times \overline{\mathbb{C}}^n$ for every $k \geq 3$. By analogy with this we conclude the proof. \square

Proof of Theorem 3. Suppose \mathcal{F} is a singular holomorphic foliation on $\overline{\mathbb{C}} \times \mathbb{C}P(2)$ given by a polynomial vector field $X(x, y, z) = P(x, y, z)\frac{\partial}{\partial x} + Q(x, y, z)\frac{\partial}{\partial y} + R(x, y, z)\frac{\partial}{\partial z}$ in affine coordinates $(x, y, z) \in \mathbb{C} \times \mathbb{C}^2 \subset \overline{\mathbb{C}} \times \mathbb{C}P(2)$. and transverse to almost every fiber of η . Let $\{x_0\} \times \mathbb{C}P(2)$ be a fiber transverse to \mathcal{F} . Then \mathcal{F} is transverse to $\{x\} \times \mathbb{C}P(2)$, $\forall x$ in a neighborhood of x_0 . This implies $P(x, y, z) = p(x)$, $\forall (x, y, z) \in \mathbb{C} \times \mathbb{C}^2$. On the other hand,

$$X(x, u, v) = p(x)\frac{\partial}{\partial x} - u^2Q(x, \frac{1}{u}, \frac{v}{u})\frac{\partial}{\partial u} + (uR(x, \frac{1}{u}, \frac{v}{u}) - uvQ(x, \frac{1}{u}, \frac{v}{u}))\frac{\partial}{\partial v}.$$

in affine coordinates $(x, u, v) \in \mathbb{C} \times \mathbb{C}^2 \hookrightarrow \overline{\mathbb{C}} \times \mathbb{C}P(2)$ with $u = \frac{1}{y}$ and $v = \frac{z}{y}$. Therefore

$$X(x, u, v) = p(x)\frac{\partial}{\partial x} - u^2\frac{1}{u^\alpha}\tilde{Q}(x, u, v)\frac{\partial}{\partial u} + (u\frac{1}{u^\beta}\tilde{R}(x, u, v) - uv\frac{1}{u^\alpha}\tilde{Q}(x, u, v))\frac{\partial}{\partial v}$$

where \tilde{Q}, \tilde{R} are polynomials in $\mathbb{C}[x, u, v]$, $\alpha = \max\{m + n; Q(x, y, z) = \sum_{l,m,n} q_{l,m,n}x^l y^m z^n\}$ and $\beta = \max\{m + n; R(x, y, z) = \sum_{l,m,n} r_{l,m,n}x^l y^m z^n\}$. If $\{x_1\} \times \mathbb{C}P(2)$ is a fiber transverse to \mathcal{F} , then $\beta \leq \alpha$. In fact, if $\beta > \alpha$, then

$$X(x, u, v) = p(x)\frac{\partial}{\partial x} - u^2\frac{1}{u^\alpha}\tilde{Q}(x, u, v)\frac{\partial}{\partial u} + \frac{1}{u^{\beta-1}}(\tilde{R}(x, u, v) - u^{\beta-\alpha}v\tilde{Q}(x, u, v))\frac{\partial}{\partial v},$$

and we recall that $\beta > \alpha$ implies $\beta > 1$. We multiply the vector field X by $u^{\beta-1}$ to obtain

$$X(x, u, v) = u^{\beta-1}p(x)\frac{\partial}{\partial x} - uu^{\beta-\alpha}\tilde{Q}(x, u, v)\frac{\partial}{\partial u} + (\tilde{R}(x, u, v) - u^{\beta-\alpha}v\tilde{Q}(x, u, v))\frac{\partial}{\partial v}.$$

We observe that X is not transverse to $\{x_1\} \times \mathbb{C}^2$ at the point $(x_1, 0, v_0)$, because if $q(v) = \tilde{R}(x_1, 0, v)$ is the zero polynomial in $\mathbb{C}[v]$, then $(x_1, 0, v_0)$ is a singularity of X . Otherwise $q(v)$ is a polynomial in $\mathbb{C}[v] \setminus \{0\}$ and the point $(x_1, 0, v_0)$ is a singularity of X or X is tangent to $\{x_1\} \times \mathbb{C}P(2)$ at this point if v_0 is a zero of $q(v)$ or it is not. Therefore $\beta \leq \alpha$, and

$$X = p(x)\frac{\partial}{\partial x} - u^2\frac{1}{u^\alpha}\tilde{Q}(x, u, v)\frac{\partial}{\partial u} + \frac{1}{u^\alpha}(uu^{\alpha-\beta}\frac{1}{u^\beta}\tilde{R}(x, u, v) - uv\tilde{Q}(x, u, v))\frac{\partial}{\partial v}.$$

in affine coordinates $(x, u, v) \in \mathbb{C} \times \mathbb{C}^2 \hookrightarrow \overline{\mathbb{C}} \times \mathbb{C}P(2)$. Notice that $\alpha \leq 2$. In fact, if $\alpha > 2$, that is, there is $k \in \mathbb{Z} \setminus \{-1, -2, \dots\}$ such that $\alpha = 3 + k$. Then

$$X(x, u, v) = p(x)\frac{\partial}{\partial x} - \frac{1}{u^{k+1}}\tilde{Q}(x, u, v)\frac{\partial}{\partial u} + \frac{1}{u^{k+2}}(u^{\alpha-\beta}\tilde{R}(x, u, v) - v\tilde{Q}(x, u, v))\frac{\partial}{\partial v}$$

and we multiply it by u^{k+1}

$$X(x, u, v) = u^{k+1}p(x)\frac{\partial}{\partial x} - \tilde{Q}(x, u, v)\frac{\partial}{\partial u} + \frac{1}{u}(u^{\alpha-\beta}\tilde{R}(x, u, v) - v\tilde{Q}(x, u, v))\frac{\partial}{\partial v}.$$

We obtain two possibilities: the polynomial S in $\mathbb{C}[x, u, v]$ defined by

$$S(x, u, v) = u^{\alpha-\beta}\tilde{R}(x, u, v) - v\tilde{Q}(x, u, v)$$

is multiple of u or it is not.

Case (i). If S is multiple of u , then \mathcal{F} is given by the holomorphic vector field

$$X(x, u, v) = u^{k+1}p(x)\frac{\partial}{\partial x} - \tilde{Q}(x, u, v)\frac{\partial}{\partial u} + (u^{\alpha-\beta}\tilde{R}(x, u, v) - v\tilde{Q}(x, u, v))\frac{\partial}{\partial v}.$$

and observe that X is not transverse to $\{x_1\} \times \mathbb{C}P(2)$ in $(x_1, 0, v_0)$.

Case (ii). If S is not multiple of u , then \mathcal{F} is given by the vector field without poles

$$X(x, u, v) = u^{k+2}p(x)\frac{\partial}{\partial x} - u\tilde{Q}(x, u, v)\frac{\partial}{\partial u} + S(x, u, v)\frac{\partial}{\partial v}$$

and observe that X is not transverse to $\{x_1\} \times \mathbb{C}P(2)$ at $(x_1, 0, v_0)$, either. Then it is a contradiction in both cases. Recall that $\beta \leq \alpha \leq 2$. We obtain the following possibilities:

Possibility 1. If $\alpha = \beta = 0$, then

$$X(x, y, z) = p(x)\frac{\partial}{\partial x} + q(x)\frac{\partial}{\partial y} + r(x)\frac{\partial}{\partial z}. \quad (15)$$

in affine coordinates $(x, y, z) \in \mathbb{C}^3 \hookrightarrow \overline{\mathbb{C}} \times \mathbb{C}P(2)$.

Possibility 2. If $\alpha = 1$, $\beta = 0$, then

$$X(x, y, z) = p(x)\frac{\partial}{\partial x} + (A(x) + B(x)y + C(x)z)\frac{\partial}{\partial y} + r(x)\frac{\partial}{\partial z}, \quad (16)$$

where $A, B, C \in \mathbb{C}[x]$.

Possibility 3. Se $\alpha = \beta = 1$, then

$$X(x, y, z) = p(x)\frac{\partial}{\partial x} + (A(x) + B(x)y + C(x)z)\frac{\partial}{\partial y} + (a(x) + b(x)y + c(x)z)\frac{\partial}{\partial z}, \quad (17)$$

where $a, A, b, B, c, C \in \mathbb{C}[x]$.

Observe that the foliation \mathcal{F} given by the vector field X defined by equation 17 (15 or 16 respectively) is transverse to almost every fiber of η . In fact, we obtain

$$X(x, u, v) = p(x)\frac{\partial}{\partial x} - u(A(x)u + C(x)v + B(x))\frac{\partial}{\partial u} + \tilde{R}(x, u, v)\frac{\partial}{\partial v},$$

in affine coordinates $(x, u, v) \in \mathbb{C}^3 \hookrightarrow \overline{\mathbb{C}} \times \mathbb{C}P(2)$ where $\tilde{R}(x, u, v) = a(x)u + c(x)v + b(x) - v(A(x)u + C(x)v + B(x))$ and so \mathcal{F} is transverse to $\{x\} \times \mathbb{C}P(2)$ if $p(x) \neq 0$. On the other hand, we obtain

$$X(x, t, s) = p(x)\frac{\partial}{\partial x} + \tilde{Q}(x, t, s)\frac{\partial}{\partial t} - s(a(x)s + b(x)t + c(x))\frac{\partial}{\partial s}$$

with $\tilde{Q}(x, t, s) = A(x)s + B(x)t + C(x) - t(a(x)s + b(x)t + c(x))$ in affine coordinates $(x, t, s) \in \mathbb{C}^3 \hookrightarrow \mathbb{C} \times \mathbb{C}P(2)$ where $s = \frac{1}{z}$ and $t = \frac{y}{z}$. In this case \mathcal{F} is transverse to $\{x\} \times \mathbb{C}P(2)$ if $p(x) \neq 0$.

Possibility 4. If $\alpha = 2$, $\beta = 0$, then

$$X(x, y, z) = p(x)\frac{\partial}{\partial x} + Q(x, y, z)\frac{\partial}{\partial y} + r(x)\frac{\partial}{\partial z}, \quad (18)$$

where $Q(x, y, z) = A(x) + B(x)y + C(x)z + D(x)yz + E(x)y^2 + F(x)z^2$ and $A, B, C, D, E, F \in \mathbb{C}[x]$.

We obtain

$$X(x, u, v) = p(x) \frac{\partial}{\partial x} - \tilde{Q}(x, u, v) \frac{\partial}{\partial u} + (u r(x) - \frac{v}{u} \tilde{Q}(x, u, v)) \frac{\partial}{\partial v},$$

where $\tilde{Q}(x, u, v) = A(x)u^2 + F(x)v^2 + C(x)uv + D(x)v + B(x)u + E(x)$. Recall that D, E or F is not the zero polynomial. Then \tilde{Q} is not multiple of u . Moreover the polynomial $S(x, u, v) = u^2 r(x) - v \tilde{Q}(x, u, v)$ is not the zero polynomial. We multiply the vector field X by u and obtain

$$X(x, u, v) = u p(x) \frac{\partial}{\partial x} - u \tilde{Q}(x, u, v) \frac{\partial}{\partial u} + S(x, u, v) \frac{\partial}{\partial v}.$$

Notice that X is not transverse to $\{x_1\} \times \mathbb{C}^2$ in $(x_1, 0, v_0)$. This is a contradiction. Thus possibility 4 does not occur.

Possibility 5. If $\alpha = 2, \beta = 1$, then

$$X(x, y, z) = p(x) \frac{\partial}{\partial x} + Q(x, y, z) \frac{\partial}{\partial y} + R(x, y, z) \frac{\partial}{\partial z}, \quad (19)$$

where $Q(x, y, z) = A(x) + B(x)y + C(x)z + D(x)yz + E(x)y^2 + F(x)z^2$, $R(x, y, z) = a(x) + b(x)y + c(x)z$ e $a, b, c, A, B, C, D, E, F \in \mathbb{C}[x]$. Then

$$X(x, u, v) = p(x) \frac{\partial}{\partial x} - \tilde{Q}(x, u, v) \frac{\partial}{\partial u} + \frac{1}{u}(u \tilde{R}(x, u, v) - v \tilde{Q}(x, u, v)) \frac{\partial}{\partial v},$$

where $\tilde{Q}(x, u, v) = A(x)u^2 + F(x)v^2 + C(x)uv + D(x)v + B(x)u + E(x)$ e $\tilde{R} = a(x)u + c(x)v + b(x)$, in affine coordinates $(x, u, v) \in \mathbb{C} \times \mathbb{C}^2 \hookrightarrow \overline{\mathbb{C}} \times \mathbb{C}P(2)$. Observe that D, E or F is not the zero polynomial. Therefore \tilde{Q} is not multiple of u , and the polynomial $S(x, u, v) = u \tilde{R}(x, y, z) - v \tilde{Q}(x, u, v)$ is not the zero polynomial. We multiply X by u and obtain

$$X(x, u, v) = u p(x) \frac{\partial}{\partial x} - u \tilde{Q}(x, u, v) \frac{\partial}{\partial u} + S(x, u, v) \frac{\partial}{\partial v},$$

and observe that X is not transverse to $\{x_1\} \times \mathbb{C}^2$ in $(x_1, 0, v_0)$. Then it is a contradiction and possibility 5 does not occur.

Possibility 6. If $\alpha = \beta = 2$, then

$$X(x, y, z) = p(x) \frac{\partial}{\partial x} + Q(x, y, z) \frac{\partial}{\partial y} + R(x, y, z) \frac{\partial}{\partial z}, \quad (20)$$

where $Q(x, y, z) = A(x) + B(x)y + C(x)z + D(x)yz + E(x)y^2 + F(x)z^2$, $R(x, y, z) = a(x) + b(x)y + c(x)z + d(x)yz + e(x)y^2 + f(x)z^2$ and $a, b, c, d, e, f, A, B, C, D, E, F \in \mathbb{C}[x]$. Then

$$X(x, u, v) = p(x) \frac{\partial}{\partial x} - \tilde{Q}(x, u, v) \frac{\partial}{\partial u} + \frac{1}{u}(\tilde{R}(x, u, v) - v \tilde{Q}(x, u, v)) \frac{\partial}{\partial v},$$

in affine coordinates $(x, u, v) \in \mathbb{C} \times \mathbb{C}^2 \hookrightarrow \overline{\mathbb{C}} \times \mathbb{C}P(2)$ where $\tilde{Q}(x, u, v) = A(x)u^2 + F(x)v^2 + C(x)uv + D(x)v + B(x)u + E(x)$ and $\tilde{R} = a(x)u^2 + f(x)v^2 + c(x)uv + d(x)v + b(x)u + e(x)$. Notice

that none of the polynomials D, E or $F \in \mathbb{C}[x]$ is identically zero and the same holds for the polynomials d, e or $f \in \mathbb{C}[x]$. Therefore the polynomial $S(x, u, v) = \tilde{R}(x, u, v) - v \tilde{Q}(x, u, v)$ is a multiple of u if and only if $e = 0$, $d = E$, $f = D$ and $F = 0$. Then the foliation \mathcal{F} is given by X is transverse to $\{x\} \times \mathbb{C} P(2)$ if $p(x) \neq 0$.

On the other hand

$$X(x, t, s) = p(x) \frac{\partial}{\partial x} + \frac{1}{s} (\tilde{Q}(x, t, s) - t \tilde{R}(x, t, s)) \frac{\partial}{\partial t} - \tilde{Q}(x, t, s) \frac{\partial}{\partial s},$$

in affine coordinates $(x, t, s) \in \mathbb{C} \times \mathbb{C}^2 \hookrightarrow \overline{\mathbb{C}} \times \mathbb{C} P(2)$ with $t = \frac{y}{z}$ e $s = \frac{1}{z}$ where

$$\tilde{Q}(x, t, s) = A(x)s^2 + E(x)t^2 + B(x)ts + C(x)s + D(x)t$$

and

$$\tilde{R}(x, t, s) = a(x)s^2 + b(x)ts + c(x)s + E(x)t + D(x).$$

Therefore

$$\tilde{Q}(x, t, s) - t \tilde{R}(x, t, s) = -ts^2 a(x) - t^2 sb(x) + ts(B - c)(x) + s^2 A(x) + sC(x),$$

and the foliation \mathcal{F} is given by the vector field without poles

$$X(x, t, s) = p(x) \frac{\partial}{\partial x} + U(x, t, s) \frac{\partial}{\partial t} - \tilde{Q}(x, t, s) \frac{\partial}{\partial s},$$

with $U(x, t, s) = -ts a(x) - t^2 b(x) + t(B - c)(x) + s A(x) + C(x)$, and it is transverse to $\{x\} \times \mathbb{C} P(2)$ if $p(x) \neq 0$.

If S is not multiple of u , then the foliation \mathcal{F} is given by the vector field without poles

$$X(x, u, v) = u p(x) \frac{\partial}{\partial x} - u \tilde{Q}(x, u, v) \frac{\partial}{\partial u} + S(x, u, v) \frac{\partial}{\partial v},$$

and it is not transverse to $\{x_1\} \times \mathbb{C}^2$ in $(x_1, 0, v_0)$. We obtain a contradiction. This ends the proof of Theorem 3. \square

The above proof indeed gives:

Corollary 2. *Let \mathcal{F} be a singular holomorphic foliation on $\overline{\mathbb{C}} \times \mathbb{C} P(2)$ given by a polynomial vector field X in affine coordinates on $\mathbb{C} \times \mathbb{C}^2$. Suppose \mathcal{F} is transverse at least one fiber $\{x_0\} \times \mathbb{C} P(2)$ of η . Then \mathcal{F} is a Riccati foliation on $\overline{\mathbb{C}} \times \mathbb{C} P(2)$.*

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